The Casimir force between parallel plates in Randall-Sundrum I model

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Abstract

The Casimir effect for parallel plates within the frame of five-dimensional Randall-Sundrum model with two branes is reexamined. We argue that the nature of Casimir force is repulsive if the distance between the plates is not extremely tiny, which is not consistent with the experimental phenomena, meaning that the Randall-Sundrum I model can not be acceptable. We also point out that the estimation of the separation between the two branes, by means of Casimir effect for two-parallel-plate system, is not feasible, in contrast to another recent study.

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The high-dimensional spacetime theory suggesting that our observalbe four-dimensional world is a subspace of a higher dimensional spacetime has a long tradition that was put forward by Kaluza and Klein more than 80 years ago [1, 2]. The high-dimensional spacetime models including the dimensionality and the geometric characteristics of extra dimensions are necessary. The main motivation for such approaches are to unify the fundamental interactions in nature. The approaches with additional dimensions were invoked for providing a breakthrough of cosmological constant and the hierarchy problems [3-8]. The issues of high-dimensional spacetimes have their own compactification and properties of extra dimensions. More theories need developing and to be realized within the frame with extra dimensions. In the Kaluza-Klein theory, one extra dimension in our universe was introduced to be compactified to unify gravity and classical electrodynamics. The quantum gravity such as string theories or braneworld scenario is developed to reconcile the quantum mechanics and gravity with the help of introducing seven extra spatial dimensions. The new approaches propose that the geometry of the extra spatial dimensions is responsible for the hierarchy problem. At first, the large extra dimensions (LED) were introduced [6-8]. In this model the additional dimensions are flat and of equal size and the radius of a toroid is limited while the size of extra space can not be too small, or the hierarchy problem remains. Another model with warped extra dimensions was also put forward [9, 10]. A four-dimensional theory compactified on a S^1/Z_2 manifold with bulk boundary cosmological constants leading to a stable four-dimensional low energy effective theory, named Randall-Sundrum (RS) models, suggested that the compact extra dimension with large curvatures to explain the reason why the large gap between the Planck and the electroweak scales exists. In RSI, one of the RS models, there are two 3-branes with equal opposite tensions and they are localized at $y = 0$ and $y = L$ respectively, with Z_2 symmetry $y \longleftrightarrow -y$, $L + y \longleftrightarrow L - y$. The Randall-Sundrum model becomes RSII when one brane is located at infinity, like $L \longrightarrow \infty$. The standard modelfields and gauge fields live on the negative tension brane which is visible, while the positive tension brane with a fundamental scale M_{RS} is hidden. Employing the additional compactified dimensions and additional warped dimension can help us to unify the interactions and resolve the hierarchy problem respectively.

The Casimir effect depends on the dimensionality and topology of the spacetime [11-20] and has received a great deal of attention within spacetime models including additional spatial dimensions. There exists strong influence from the possibility of the existence, the size and the geometry of extra dimensions on the Casimir effect, the evaluation of the vacuum zero-point energy. The precision of the measurements of the attraction force between parallel plates as well as other geometries has been greatly improved practically [21-24], leading the Casimir effect to be a remarkable observable and trustworthy consequence of the existence of quantum fluctuations. The experimental results clearly show that the attractive Casimir force between the parallel plates vanishes when the plates move apart from each other to the very distant place. In particular it must be pointed out that no repulsive force appears in this case. Therefore the Casimir effect for parallel plates can become a window to probe the high-dimensional Universe and can be used to research on a large class of

related topics on the various models of spaccetimes with more than four dimensions. More efforts have been made on the studies. Within the frames of several kinds of spacetimes with high dimensionality the Casimir effect for various systems has been discussed. The electromagnetic Casimir effect for parallel plates in a high-dimensional spacetime has been studied and the subtraction of the divergences in the Casimir energy at the boundaries is realized [25, 26]. Some topics were studied in the high-dimensional spacetime described by Kaluza-Klein theory. It is shown analytically that the extra-dimension corrections to the Casimir effect for a rectangular cavity in the presence of a compactified universal extra dimension are very manifest [27]. It was also proved rigorously that there will appear repulsive Casimir force between the parallel plates when the plates distance is sufficiently large in the spacetime with compactified additional dimensions, and the higher the dimensionality is, the greater the repulsive force is, unless the Casimir energy outside the system consisting of two parallel plates is considered [27-37]. The research on the Casimir energy within the frame of Kaluza-Klein theory to explain the dark energy has been performed and is also fundamental, and a lot of progress have been made [38-40]. In the context of string theory the Casimir effect was also investigated [41-44]. Also in the Randall-Sundrum model, the Casimir effect has been investigated to stabilize the distance between branes (radon) [45-49]. In particular the evaluation of the Casimir force between two parallel plates under Dirichlet conditions has been performed in the Randall-Sundrum models with one extra dimension [50-52]. The upper limit on the separation between two branes $kR \leq 20$ if the curvature parameter k of five-dimensional anti de Sitter spacetime (AdS_5) is equal to the Planck scale has been obtained in RSI model. It was shown that the required value for solving the hierarchy problem is $kR \sim 12$ and does not conflict with the results like $kR \leq 20$ from the Casimir effect. In the case of RSII model, the influence from extra dimension on the Casimir force between parallel plates is so small that it can be neglected. The Casimir effect is an efficient tool to explore the high-dimensional spacetime.

It is fundamental and significant to re-examine the Casimir effect for two parallel plates in the Randall-Sundrum I models. Here we argue that the Casimir force between the two parallel plates could be repulsive in the five-dimensional RSI background and the repulsive force results disfavoured by the experimental evidence will contradict the conclusions from Ref. [50]. Certainly the RSI model can not be accepted. The main purpose of this paper is to scrutinize the two-parallel-plate Casimir force in the RSI model again in order to confirm the nature of the Casimir force, comparing with the clear and definite experimental results. We obtain the Casimir force within the RSI model by means of the zeta-function regularization and discuss the dependence of the force on the plates gap and the separation of the two branes. Our conclusions are emphasized in the end.

In this study, we investigate a massless scalar field living in the bulk in the five-dimensional RSI model of spacetime. Within the frame the spacetime metric is chosen as,

$$ds^2 = e^{-2k|y|}g_{\mu\nu}dx^\mu dx^\nu - dy^2 \tag{1}$$

where the variable k assumed to be of the order of the Planck scale governs the degree of curvature of the AdS_5 with constant negative curvature. That the extra dimension is compactified on an orbifold gives rise to the generation of the absolute value of y in the metric. We follow the procedure of Ref. [50, 52]. In the five-dimensional spacetime with the background metric denoted in Eq.(1), the equation of motion for a massless bulk scalar field Φ is,

$$g^{\mu\nu}\partial_\mu\partial_\nu\Phi + e^{2ky}\partial_y(e^{-4ky}\partial_y) = 0 \tag{2}$$

where $g^{\mu\nu}$ is the usual four-dimensional flat metric with signature -2. The field confining between the two parallel plates satisfies the Dirichlet boundary conditions $\Phi(x^\mu, y)|_{\partial\Omega} = 0$, $\partial\Omega$ positions of the plates in coordinates $\mathbf{x}$. We can choose the $y-$dependent part of the field $\Phi(x^\mu, y)$ as $\chi^{(N)}(y)$ in virtue of separation of variables. Having solved the equation of motion of $\chi^{(N)}(y)$ we obtain their general expression for the nonzero modes in terms of Bessel functions of the first and second kind as,

$$\chi^{(N\neq 0)}(y) = e^{2ky}(a_1 J_2(\frac{m_N e^{ky}}{k}) + a_2 Y_2(\frac{m_N e^{ky}}{k})) \tag{3}$$

where a_1 and a_2 are arbitrary constants. The effective mass term for the scalar field denoted as m_N can also be obtained by means of integration out the fifth dimension y. In the case of RSI model, the hidden and visible 3-branes are located at $y = 0$ and $y = \pi R$ respectively, which leads the Neumann boundary conditions $\frac{\partial\chi^{(N)}}{\partial y}|_{y=0} = \frac{\partial\chi^{(N)}}{\partial y}|_{y=\pi R} = 0$, so a general reduced equation reads,

$$m_N \approx e^{-\pi kR}(N + \frac{1}{4})k\pi = \kappa(N + \frac{1}{4}) \tag{4}$$

where

$$\kappa = \pi k e^{-\pi kR} \tag{5}$$

by means of the asymptotic form of the Bessel function of the first kind $J_\nu(x)$ and here we assume $N \gg 1$ or equivalently $\pi kR \gg 1$ throughout our work. we must point out that the effective mass term for the scalar field m_N with the integration out the fifth dimension y holds $m_N = 0$ when $N = 0$ because the first zero of the Bessel function $J_1(x)$ vanishes.

The models of the vacuum for parallel plates under the Dirichlet and modified Neumann boundary conditions as mentioned above in RSI model can be expressed as,

$$\omega_{nN} = \sqrt{\mathbf{k}_\perp^2 + (\frac{n\pi}{a})^2 + m_N^2} \tag{6}$$

where

$$\mathbf{k}_\perp^2 = \mathbf{k}_1^2 + \mathbf{k}_2^2 \tag{7}$$

here $\mathbf{k}_1$ and $\mathbf{k}_2$ are the wave vectors in directions of the unbound space coordinates parallel to the plates surface and a is the distance of the plates. Here n and N represent positive integer.

Therefore the total energy density of the fields in the interior of the system involving two parallel plates in the RSI model reads,

$$\varepsilon = \int d^2\mathbf{k}_\perp \sum_{n,N=0}^{\infty'} \omega_{nN}$$
$$= -\frac{\sqrt{\pi}}{4}\Gamma(-\frac{3}{2})\kappa^3[E_2(-\frac{3}{2};\frac{\pi^2}{\kappa^2 a^2},1;0,\frac{1}{4}) - (\frac{\pi}{\kappa a})^3 E_1(-\frac{3}{2};1;\frac{\kappa a}{4\pi}) + \frac{\pi^3}{a^3}\zeta(-3) - \frac{1}{64}] \qquad (8)$$

where the prime means that the term with $n = N = 0$ is excluded and the zeta functions of Epstein-Hurwitz type are defined by,

$$E_p(s; a_1, a_2, \cdots, a_p; c_1, c_2, \cdots, c_p)$$
$$= \sum_{\{n_j\}=0}^{\infty}(\sum_{j=1}^{p} a_j(n_j + c_j)^2)^{-s} \qquad (9)$$

and

$$E_1(s; a; c) = \sum_{n=1}^{\infty}(an^2 + c)^{-s} \qquad (10)$$

here $\{n_j\}$ stands for a short notation of $n_1, n_2, \cdots, n_p$, n_j a nonnegative integer, and $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ is Riemann zeta function and $\zeta_H(s,q) = \sum_{n=0}^{\infty}(n+q)^{-s}$ is the Hurwitz zeta function. Following the approach of Ref.[50], we regularize the Eq.(8) to obtain the Casimir energy density of parallel plates in the five-dimensional background governed by RSI model as follow,

$$\varepsilon_C = \frac{\pi}{384}\kappa^3 - \frac{\pi^4}{360}\frac{\kappa^3}{\mu^3} + \frac{\pi}{32}\frac{\kappa^3}{\mu}\sum_{n=1}^{\infty}n^{-2}K_2(\frac{\mu}{2\sqrt{\pi}}n)$$
$$-\frac{1}{2}\frac{\kappa^3}{\mu}\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty}n_1^{-2}(n_2 + \frac{1}{4})^2 K_2(2\mu n_1(n_2 + \frac{1}{4})) \qquad (11)$$

and

$$\mu = \kappa a = \pi k a e^{-\pi k R} \qquad (12)$$

and here $K_\nu(z)$ is the modified Bessel function of the second kind. The terms with series converge very quickly and only the first several summands need to be taken into account for numerical calculation to further discussion. We discuss the Casimir energy density in the limiting case that the plates separation a is large enough,

$$\varepsilon_C(\mu \gg 1) = \frac{\pi}{6}\kappa^3\zeta_H(-3,\frac{1}{4}) - \frac{\pi}{384}\kappa^3 < 0 \qquad (13)$$

which means that the sign of the Casimir energy keeps negative. Certainly we continue focusing on the Casimir force which is given by the derivative of the Casimir energy with respect to the plate

distance. According to the Casimir energy density denoted as (11), the Casimir force per unit area of plates becomes,

$$
\begin{aligned}
f_C &= -\frac{\partial \varepsilon_C}{\partial a} \\
&= -\frac{1}{2}\frac{\kappa^4}{\mu^2}\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty} n_1^{-2}(n_2+\frac{1}{4})^2 K_2(2\mu n_1(n_2+\frac{1}{4})) \\
&\quad -\frac{1}{2}\frac{\kappa^4}{\mu}\sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty} n_1^{-1}(n_2+\frac{1}{4})^3[K_1(2\mu n_1(n_2+\frac{1}{4}))+K_3(2\mu n_1(n_2+\frac{1}{4}))] \\
&\quad +\frac{\pi}{32}\frac{\kappa^4}{\mu^2}\sum_{n=1}^{\infty} n^{-2} K_2(\frac{\mu}{2\sqrt{\pi}}n)+\frac{\sqrt{\pi}}{128}\frac{\kappa^4}{\mu}\sum_{n=1}^{\infty} n^{-1}[K_1(\frac{\mu}{2\sqrt{\pi}}n)+K_3(\frac{\mu}{2\sqrt{\pi}}n)] \\
&\quad -\frac{\pi^4}{1080}\frac{\kappa^4}{\mu^4}
\end{aligned}
\tag{14}
$$

It is enough to make the first several summands because of quickly convergent series in the expression. If the dimensionless variable $\mu \gg 1$, which means that the gap between two plates is much larger than the separation between the two branes in the RSI model, the Casimir force vanishes,

$$
\lim_{\mu \longrightarrow \infty} f_C = 0
\tag{15}
$$

which is not in conflict with the experimental evidence. We have to perform the burden and surprisingly difficult calculation in order to scrutinize the nature of the two-parallel-plate Casimir force in the RSI model within a wider region. The dependence of the Casimir force on the variable $\mu = \kappa a$ is potted in the Figure 1. We calculate the Casimir force expression (14) to find that there must exist a special value of variable μ denoted as $\mu_0 = 0.156$. When $\mu < \mu_0$, the sign of the Casimir force between the two parallel plates is negative, which means that the plates attract each other. When the plates separation is sufficiently large to keep $\mu > \mu_0$, the nature of the Casimir force is repulsive. Although the Casimir force is equal to the zero as the distance between the plates approaches to the infinity, it should be pointed out that the force keeps repulsive during the process as $\mu > \mu_0$. It is clear that our results are different from those of Ref. [50]. The results that the Casimir force within the system containing two parallel plates is repulsive are not consistent with the experimental phenomenon. In this system no repulsive Casimir force appears according to the measurements. It should also be emphasized that the value of special parameter μ_0 can be different for different kinds of fields referring to more complicated boundary conditions than the case of scalar field we consider here, but the repulsive Casimir force must generate as the plates are sufficiently far away from each other. It is necessary to make some estimations for the sake of comparison. In Ref. [50] the Casimir force was plotted within the range of plate separation mainly from $0.5 \times 10^{-6}m$ to $2 \times 10^{-6}m$ by means of comparison of the Casimir force in RSI model with the ones in the case of the standard Casimir force supported by measurements without RS contribution to obtain the upper bound $kR \leq 20$ while the AdS_5 curvature scale k is set to the $10^{16}GeV$ or Planck scale $10^{19}GeV$. In Ref. [50] the Casimir force was drawn in Figures when the

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plates distance belongs to the range mainly from $0.5 \times 10^{-6} m$ to $3 \times 10^{-6} m$. Having substituted the results from Ref. [50] like upper bound $kR \leq 20$ and the values of the AdS_5 curvature scale k and plate distance like $0.5 \times 10^{-6} m \leq \mu \leq 3\mu m$ which were employed by Ref. [50] into the relation between the plate distance and radion denoted in (12), we give rise to the range of the dimensionless variable $\mu \in [41.077, 164.308]$. Certainly the values of dimensionless variable in the case of Ref. [50] are much larger than the special value μ_0 that we discover above and the Casimir force should be positive which lead the two parallel plates to move apart. In addition we research on the restriction on the plate gap when we set $kR \sim 12$ the required value for solving the hierarchy problem and k to the Planck scale. According to the definition of μ like (12) and the special value $\mu_0 = 0.156$, we find that the Casimir force between plates is attractive only when the restriction on plate separation is $a < 2.2 \times 10^{-20} m$ or the force becomes repulsive, which is not consistent with the experimental results. It should be pointed out that the equation is valid asymptotically for $N \gg 1$ although the reduced equation (4) for the effective mass of the scalar bulk field is expressed as an approximation. According to the properties of Bessel functions of the first and second kind, the error is about 3% when $N = 1$ and the error is 0.3% and 0.1% for $N = 2$ and $N = 3$ respectively, etc., displaying that the error drops very quickly with increasing N. The deviation from the approximation can not change our conclusion, so the repulsive Casimir force denoted as positive magnitude of f_C will appear inevitably when the plates distance is sufficiently large.

In conclusion, there must appear the repulsive force between two parallel plates in the five-dimensional Randall-Sundrum model with two branes if the plate separation is not extremely tiny and the results that the Casimir force is repulsive conflict with the experimental evidence. We come to a different conclusion from those of Ref. [50]. Having discussed the Casimir force between parallel plates in the frame of RSI model in detail, we reveal that the Casimir force always remains repulsive as the plate separation is larger than a very small quantity which is equal to $10^{-20} m$ approximately while $kR \sim 12$, the required value for solving the hierarchy problem and k is set to be the Planck scale, but the experimental evidence confirms that no repulsive Casimir force appears in this case. Of course it is impossible to estimate the branes distance of RSI model by means of Casimir force for standard two-parallel-plate device. Although we are limited here in the case of massless scalar field obeying the Dirichlet boundary condition for simplicity and comparison, the special variable μ_0 will be different for different fields with different kinds of boundary conditions, but the special parameter μ_0 must exist. If the dimensionless variable μ showing the relation between the separations of plates and branes is larger than the special value μ_0 which is very small, the Casimir force must be repulsive which is excluded by the experiment, which mean that the Randall-Sundrum I model can not be acceptable. The topics about the Casimir force in the five-dimensional Randall-Sundrum model needs to be developed further and related topics also need further research.

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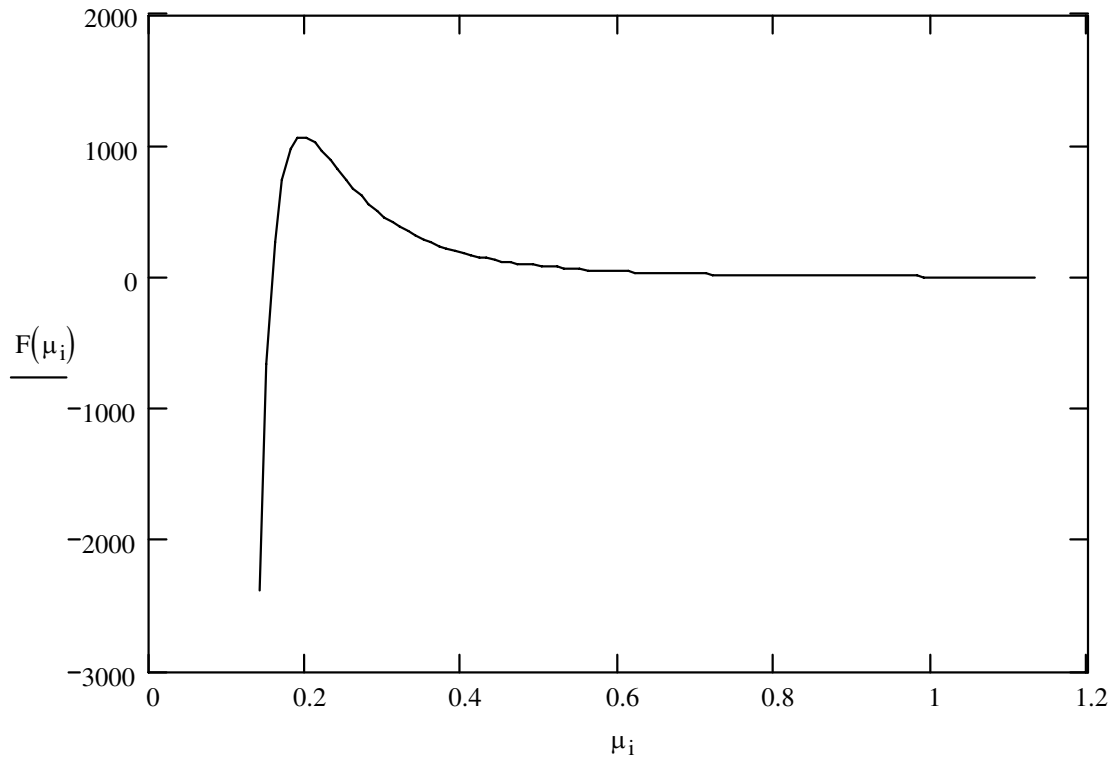
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Figure 1: The Casimir force per unit area in unit of κ^4 between parallel plates versus the dimensionless variable denoted as $\mu = \kappa a$